Intergrerend project dynamische systemen

Thursday 24 June 2011, 9:00-10:30

university of groningen

Exam Problem Sheet

The exam consists of 4 problems. You may answer in Dutch or in English. You can achieve 50 points in total.

1. [**5+5 Points.**]

   Consider the growth model

   $$x' = x - h(1 + \sin t)\,,$$

   where x denote the size of a population and $-h(1 + \sin t)$ is a periodic harvesting term.

   (a) Determine the general solution of this system, and show that there is exactly one periodic solution. What is the condition on h, and what is the biological interpretation of this condition?

   (b) Compute the Poincaré map for this system, and use it to verify your result from part (a) that there is exactly one periodic solution.

2. [**4+6 Points.**]

   Consider the pair of two-dimensional systems

   $$X' = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} X \text{ and } Y' = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix} Y\,.$$

   (a) Determine the phase portraits of the two systems.

   (b) Find a topological conjugacy between the two systems.

3. [**9+6 Points.**]

   (a) For each of the following bifurcations, give an example of a family of one-dimensional systems of the form $x' = f_a(x)$ where $a \in \mathbb{R}$ is a parameter such that at $a = 0$ one finds

      i. a transcritical bifurcation,

      ii. a saddle-node bifurcation, and

      iii. a pitchfork bifurcation.

      Also sketch the corresponding bifurcation diagrams.

— please turn over —

(b) Consider a first-order differential equation

$$x' = f_a(x)$$

for which $f_a(x_0) = 0$ and $f_a'(x_0) \neq 0$. Prove that the differential equation

$$x' = f_{a+\epsilon}(x)$$

has an equilibrium point $x_0(\epsilon)$ where $\epsilon \mapsto x_0(\epsilon)$ is a smooth function satisfying $x_0(0) = x_0$ for ϵ sufficiently small.

4. **[10+5 Points.]**

(a) Prove that the equilibrium at the origin $(x, y, z) = 0$ of the system

$$\begin{aligned} x' &= -x^3 \\ y' &= -y\,(x^2 + z^2 + 1) \\ z' &= -\sin z \end{aligned}$$

is asymptotically stable and determine its basin of attraction.

(b) State Lasalle's Invariance Principle.